UNIFORMIZATION OF SYMMETRIC RIEMANN SURFACES BY SCHOTTKY GROUPS(1)

BY

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1. **Introduction.** A Riemann surface S is called symmetric if there exists an anti-conformal map ϕ of S onto itself such that $\phi^2 = \text{identity}$. We say that ϕ is a symmetry on S.

The classical "retrospection theorem" asserts the existence of representations of closed Riemann surfaces of genus g by "Schottky groups," groups generated by Möbius transformations $A_1, \cdots, A_g$ such that A_i maps the exterior of Γ_i into the interior of Γ_i', where $\Gamma_1, \Gamma_1', \cdots, \Gamma_g, \Gamma_g'$ are disjoint Jordan curves bounding a $2g$-times connected domain, a standard fundamental domain for the group.

We will show that a closed *symmetric* Riemann surface of genus g can be represented by a Schottky group which has a standard fundamental domain which exhibits the symmetry. This result is contained in Theorems I, II and III of §§4-6. The proof does not use the classical theorem.

As a corollary, in §7, we obtain a new proof of the Koebe theorem: every n-times connected planar domain can be conformally mapped onto a plane domain exterior to n disjoint circles.

Techniques from the theory of quasiconformal mappings are used to obtain these results.

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2. **Quasiconformal mappings.** We recall [1], [3] that a homeomorphism $w(z)$ of a plane domain Δ onto another plane domain $\tilde{\Delta}$ is said to be *quasiconformal* if it has generalized derivatives satisfying, at each point $z \in \Delta$, a Beltrami equation $w_{\bar{z}} = \mu(z) w_z$ with $\mu(z) \in M_\Delta$, where $\mu(z) \in M_\Delta$ if it is defined and measurable in Δ and $\operatorname{ess.sup}|\mu(z)| \leqq k < 1$ for $z \in \Delta$.

For a given $\mu(z) \in M_C$ (C the complex plane) there exists a unique quasiconformal mapping $w^\mu(z)$ of C onto itself satisfying $w_{\bar{z}} = \mu(z) w_z$ and normalized by the conditions $w(0) = 0$ and $w(1) = 1$.

If $\mu(z) \in M_C$ is *compatible* with the Möbius transformation $A(z)$:

$$\mu \circ A = (A_z/\overline{A}_z)\mu$$

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or with the anti-Möbius transformation $B(z) = (a\bar{z} + b)/(c\bar{z} + d)$:

$$\mu \circ B = (B_{\bar{z}}/\overline{B_{\bar{z}}})\bar{\mu}$$

then $A^\mu = w^\mu \circ A \circ (w^\mu)^{-1}$ and $B^\mu = w^\mu \circ B \circ (w^\mu)^{-1}$ are Möbius and anti-Möbius transformations respectively. If G is a Schottky group generated by the transformations $\{A_j\}$, then the transformations $\{A_j^\mu\}$ generate a Schottky group G^μ.

We need, also, the following lemmas:

Lemma 1. *Let $M(z)$ and $N(z)$ be two anti-Möbius transformations. If $\mu(z)$ is compatible with $M(z)$ and $M \circ N(z)$ then it is compatible with $N(z)$.*

The proof is by calculation.

Lemma 2. *If $\mu(z) \in M_C$ and is compatible with the anti-Möbius transformation $R(z)$, reflection in $C_{a,\rho}$ (the circle with radius ρ and center at $z = a$), that is*

$$R(z) = a + \rho^2/\overline{z - a}$$

then $w^\mu(C_{a,\rho})$ is a circle (with center at $w^\mu(a)$) and

$$R^\mu(w^\mu(z)) = w^\mu(a) + \lambda^2/\overline{w^\mu(z) - w^\mu(a)},$$

i.e., reflection in $w(C_{a,\rho})$.

Proof. Let $\psi(z) = w^\mu(z) + \rho^2/\overline{w^\mu(R(z)) - w^\mu(a)}$. Then it is easily shown that $\psi(z) - w^\mu(a)$ and $w^\mu(z) - w^\mu(a)$ both satisfy the same Beltrami equation. They are both 0 when $z = a$ and ∞ when $z = \infty$. It follows by uniqueness that one is a multiple of the other. Then $[w^\mu(z) - w^\mu(a)]\overline{[w^\mu(R(z) - w^\mu(a)]} = \text{constant}$.

For z' on $C_{a,\rho}, R(z') = z'$ and $|w^\mu(z') - w^\mu(a)| = \lambda$ where λ is a positive constant. Hence $w^\mu(C_{a,\rho})$ is a circle with center $w^\mu(a)$ and radius λ.

Remark. If, in Lemma 2, $C_{a,\rho}$ is the unit circle, then $R^\mu = w^\mu \circ R \circ (w^\mu)^{-1}$ is again reflection in the unit circle. If, in addition, μ is compatible with the Möbius transformations A and B and $A = R \circ B \circ R$, then $A^\mu = R^\mu \circ B^\mu \circ R^\mu$. This follows by a simple calculation.

Lemma 3. *If $\mu(z) \in M_C$ and is compatible with the anti-Möbius transformation*

$$Q(z) = a - \rho^2/\overline{z - a}$$

which is reflection in the circle $C_{a,\rho}$ of radius ρ and center a, followed by a rotation about a by the angle π, then $w^\mu(C_{a,\rho})$ is a "quasicircle" (i.e., if w_1 is on $w^\mu(C_{a,\rho})$ then the line through w_1 and the "center" $b = w^\mu(a)$ intersects $w^\mu(C_{a,\rho})$ in a point w_2 such that $(w_2 - b)\overline{(w_1 - b)}$ = negative constant). The anti-Möbius transformation $Q^\mu(z)$ maps the exterior of $w^\mu(C_{a,\rho})$ into its interior in such a way that a point on $w^\mu(C_{a,\rho})$ is mapped into its "diametrically opposed" point.

Proof. As in Lemma 2, one can show that

$$[w^\mu(z) - w^\mu(a)]\overline{[w^\mu(Q(z)) - w^\mu(a)]} = \text{constant} = c.$$

Setting $z = a + \rho$ and then $z = a - \rho$ we see that c is real. Suppose $c > 0$. Then for z_0 such that $w^\mu(z_0) = w^\mu(a) + \sqrt{c}$, $w^\mu(Q(z_0)) = w^\mu(a) + \sqrt{c}$ also, so that (since w^μ is a homeomorphism) $z_0 = Q(z_0)$. But $Q(z)$ is fixed point free. Hence $c = -\lambda^2$ and

$$Q^\mu(w^\mu(z)) = w^\mu(Q(z)) = w^\mu(a) - \lambda^2/\overline{w^\mu(z) - w^\mu(a)}.$$

Suppose now that w_0 is on $w^\mu(C_{a,\rho})$. Let $w_0 = w^\mu(a) + \rho e^{i\theta}$. Then $Q^\mu(w_0) = w^\mu(a) - \lambda^2/\overline{w_0 - w^\mu(a)} = w^\mu(a) - (\lambda^2/\rho)e^{i\theta}$. But then $Q^\mu(w_0)$, which is on $w^\mu(C_{a,\rho})$, is diametrically opposed to w_0. Hence $w^\mu(C_{a,\rho})$ is a quasicircle.

We recall also [3] that if a homeomorphic map f of a compact Riemann surface S onto another compact Riemann surface S' is given, there exists a quasiconformal map $\tilde{f}$ of S onto S' (i.e., a map which is quasiconformal in terms of local parameters) which is homotopic to f. If, in addition, S and S' admit anti-conformal involutions ϕ and ϕ' and if $f \circ \phi = \phi' \circ f$, then $\tilde{f}$ may be chosen so as to satisfy the relation $\tilde{f} \circ \phi = \phi' \circ \tilde{f}$. If $\mu(z) \in M_C$ and $\mu(z)$ is compatible with the generators $\{A_i\}$ of a Schottky group G, then, denoting by L and L^μ the set of limit points of G and G^μ respectively, w^μ: $C \to C$ induces a quasiconformal map of $(C - L)/G$ onto $(C - L^\mu)/G^\mu$. Furthermore, if S, S' and S'' are three Riemann surfaces and f^μ and h^μ quasiconformal maps; f^μ: $S \to S'$ and h^μ: $S \to S''$ both satisfying (in terms of local coordinates) the same Beltrami equation on S, then $h^\mu \circ (f^\mu)^{-1}$ is a conformal map of S' onto S''.

3. **Symmetric surfaces.** If a symmetry ϕ on S leaves fixed a point of S, then it leaves fixed a closed, analytic, Jordan curve through the point, which we call a transition curve.

If the $\tau \geqq 0$ transition curves separate S, a symmetric Riemann surface of genus g, into two disjoint surfaces (orthosymmetry), we say that S is symmetric of type $(g, +\tau)$ with respect to the symmetry ϕ; otherwise (diasymmetry) it is of type $(g, -\tau)$. In the former case S/ϕ is an orientable surface with τ holes and $(g - \tau + 1)/2$ handles. In the latter case S/ϕ is a nonorientable surface with τ holes. Since, topologically, on a nonorientable surface a handle can be replaced by two cross caps, S/ϕ is homeomorphic to a surface with τ holes and, say, k cross caps. It is easily seen that $k = g - \tau + 1$. From these remarks we observe that if S is symmetric of type $(g, \epsilon\tau)$, $\epsilon = \pm 1$, then:

$$\text{(1)} \qquad \begin{aligned} &\text{if } \epsilon = +1, \text{ then } g - \tau + 1 \text{ is even and } 0 \leqq g - \tau + 1 \leqq g, \\ &\text{if } \epsilon = -1, \text{ then } 0 \leqq \tau \leqq g. \end{aligned}$$

4. **Orthosymmetric surfaces.** Given $\epsilon = \pm 1$ and integers $g > 0$ and $\tau \geqq 0$ satisfying (1), we construct a "standard model of type $(g, \epsilon\tau)$." We as-

sume at first that $\epsilon = +1$ and hence $\tau > 0$. Let C_s $(1 \leqq s \leqq \tau - 1)$ and H_r $(1 < r < g - \tau + 1)$ be g disjoint circles exterior to the unit circle C_0, and with centers (a_s and a_r respectively) on the real axis. Denote by $R_s(z)$ reflection in the circle C_s. Let $A_r(z)$ be a Möbius transformation which maps the exterior of H_r onto the interior of H_{r+1}, $r = 1, 3, \cdots, g - \tau$.

Reflect the circles $C_1, \cdots, C_{\tau-1}$ and $H_1, \cdots, H_{g-\tau+1}$ in C_0, obtaining circles $C_1', \cdots, C_{\tau-1}'$ and $H_1', \cdots, H_{g-\tau+1}'$. The exterior of these $2g$ circles we denote by F and note that F is a standard fundamental domain of the Schottky group G generated by the g Möbius transformations

$$A_1, A_3, \cdots, A_{g-\tau}, A_1', A_3', \cdots, A_{g-\tau}', R_0 \circ R_1, \cdots, R_0 \circ R_{\tau-1}$$

where $A_r'(z) = R_0 \circ A_r \circ R_0(z)$. We observe that $R_0 \circ R_s(z)$ (reflection in C_s followed by reflection in C_0) maps C_s onto C_s' in such a way that points on C_s and C_s' which are symmetrically situated with respect to C_0, are identified by $R_0 \circ R_s(z)$ and hence by the group G.

Denoting by π the canonical mapping of $(C - L)$ onto $(C - L)/G$, we see that the surface $F/G = (\boldsymbol{C} - L)/G$ is symmetric of type $(g, +\tau)$ with respect to the symmetry R, $R \circ \pi = \pi \circ R_0$. We call it the *standard model of type* $(g, +\tau)$. If it is identified under R, the resulting surface $(F/G)/R = F/\{G, R_0\}$ has, by computing the Euler characteristic, τ holes and $(g - \tau + 1)/2$ handles.

Given a symmetric surface S of type $(g, +\tau)$, there exists, therefore, a homeomorphism f: $(F/G)/R = ((C - L)/G)/R \to S/\phi$. We extend f to a map of F/G onto S by the requirement $f \circ R = \phi \circ f$. The homeomorphism f can be deformed into a quasiconformal map, which we again denote by f, of F/G onto S satisfying the same requirement.

The map f defines in F a function $\mu(z) = f_{\bar z}^*/f_z^*$ where $f^*(z) = \zeta \circ f \circ z^{-1}$ (z and ζ being local coordinates near p_0 on F/G and near $f(p_0)$ on S respectively). Due to the above requirement, $\mu(z)$ is compatible with $R_0(z)$. We extend μ to $\boldsymbol{C}$ by requiring that it be compatible with G and observe that $\mu \in M_{\boldsymbol{C}}$. Let $w^\mu(z)$ be the (unique) quasiconformal map of $\boldsymbol{C}$ onto itself satisfying $w_{\bar z} = \mu(z) w_z$ with $w(0) = 0$ and $w(1) = 1$. Denote by $\tilde w^\mu$ the induced map of $(\boldsymbol{C} - L)/G$ onto $(\boldsymbol{C} - L^\mu)/G^\mu$. It is easily seen that $F^\mu = w^\mu(F)$ is a standard fundamental domain of the Schottky group G^μ. But then $h = f \circ (\tilde w^\mu)^{-1}$ is a conformal map of $(\boldsymbol{C} - L^\mu)/G^\mu$ onto S. The Schottky group G^μ, which has the fundamental domain F^μ, therefore represents the symmetric surface S.

We examine now the fundamental domain F^μ. By the remark following Lemma 2,

(a) F^μ is symmetric with respect to reflection in the unit circle and

$$h \circ R^\mu = f \circ (\tilde w^\mu)^{-1} \circ R^\mu = f \circ R \circ (\tilde w^\mu)^{-1} \doteq \phi \circ f \circ (\tilde w^\mu)^{-1} = \phi \circ h$$

so that the symmetry ϕ in S is represented by reflection in the unit circle.

(b) $A_r^\mu(w)$ and $A_r'^\mu(w)$ map the exterior of the symmetrically situated Jordan curves H_r^μ and $H_r'^\mu$ onto the interior of the symmetrically situated Jordan curves H_{r+1}^μ and $H_{r+1}'^\mu$ respectively $(r = 1, 3, \cdots, g - \tau)$. Here, we denote by Γ^μ, the image of a curve Γ under w^μ. Furthermore

$$A_r'^\mu(w) = R_0^\mu \circ A_r^\mu \circ R_0^\mu(w).$$

(c) C_s^μ and $C_s'^\mu$ are *circles* (by Lemmas 1 and 2) and the Möbius transformation $(R_0 \circ R_s)^\mu = R_0^\mu \circ R_s^\mu$ maps the exterior of C_s^μ onto the interior of $C_s'^\mu$ in such a way that two symmetrically situated points on C_s^μ and $C_s'^\mu$ are identified under the group G^μ (specifically, by the element $R_0^\mu \circ R_s^\mu$). As a result, the points on F^μ which lie on these circles are left fixed (as is the unit circle C_0^μ) by the symmetry: reflection in C_0^μ. We summarize these results in

THEOREM I. *A symmetric surface S, of type $(g, +\tau)$ with respect to a symmetry ϕ, can be represented by a Schottky group which has a fundamental domain symmetric with respect to reflection in the unit circle C, and bounded by* (i) $\tau - 1$ *identified pairs of symmetrically situated circles* $\Gamma_1, \Gamma_1', \cdots, \Gamma_{\tau-1}, \Gamma_{\tau-1}'$ *and* (ii) $(g - \tau + 1)/2$ *identified pairs of Jordan curves in the exterior of C and* $(g - \tau + 1)/2$ *symmetrically situated identified pairs of Jordan curves in the interior of C. The symmetry ϕ on S is represented by reflection in C and the τ transition curves on S by C and the $\tau - 1$ pairs of circles* $\Gamma_1, \Gamma_1', \cdots, \Gamma_{\tau-1}, \Gamma_{\tau-1}'$.

5. **Diasymmetric surfaces with fixed points.** We now extend the results of §4 to symmetric surfaces for which $\tau \neq 0$ but $\epsilon = -1$.

To obtain the standard model of type $(g, -\tau)$ we construct g pairs of circles:

$$C_1, C_1', \cdots, C_{\tau-1}, C_{\tau-1}', K_1, K_1', \cdots, K_{g-\tau+1}, K_{g-\tau+1}'$$

symmetrically situated with respect to the unit circle C_0 as in §4. If we let $Q_i(z)$ be reflection in K_i, followed by rotation about the center b_i of K_i by the angle π, and define $R_s(z)$ as in §4 we find that F, the exterior of the $2g$ circles, is a fundamental domain of the Schottky group

$$G = \{R_0 \circ Q_1, \cdots, R_0 \circ Q_{g-\tau+1}, R_0 \circ R_1, \cdots, R_0 \circ R_{\tau-1}\}.$$

Again, under $R_0 \circ R_s$, symmetrical points on C_s and C_s' are identified and $R_0 \circ Q_i$ maps each point P of K_i onto the point of K_i' which is diametrically opposed to $R_0(P)$, the reflection of P in C_0.

F/G, the *standard model of type* $(g, -\tau)$ has genus g, is symmetric with respect to $R(z)$, and has τ transition curves. $(F/G)/R$ has τ holes and $g - \tau + 1$ holes with diametrically opposed points identified (i.e., cross caps). Then, if S is a symmetric surface of type $(g, -\tau)$, $\tau \neq 0$, there exists a homeomorphism $f\colon F/G \to S$ satisfying $f \circ R = \phi \circ f$.

The procedure of §4 can be repeated to obtain a Schottky group G^μ (with a fundamental domain F^μ) which represents S. F^μ has the properties (a) and (c) of §4 and, in addition, by Lemmas 1 and 3, (b′) K_i^μ and $K_i'^\mu$ are quasicircles and the Möbius transformation $(R_0 \circ Q_i)^\mu(z)$ maps the exterior of K_i^μ onto the interior of $K_i'^\mu$ in such a way that each point P on one quasicircle is identified with the point P' on the other quasicircle which is diametrically opposed to the reflection $R_0^\mu(P)$ of P. As a result, a point on F^μ which lies on one of the quasicircles is identified, under the group $\{G^\mu, R_0^\mu\}$, with its diametrically opposed point on the quasicircle. We state

Theorem II. *A symmetric Riemann surface S of type $(g, -\tau)$, $\tau \neq 0$, with respect to a symmetry ϕ, can be represented by a Schottky group which has a standard fundamental domain symmetric with respect to reflection in the unit circle C, and bounded by* (i) *$\tau - 1$ identified pairs of symmetrically situated circles $\Gamma_1, \Gamma_1', \cdots, \Gamma_{\tau-1}, \Gamma_{\tau-1}'$ and* (ii) *$g - \tau + 1$ identified pairs of symmetrically situated quasicircles $\Lambda_1, \Lambda_1', \cdots, \Lambda_{g-\tau+1}, \Lambda_{g-\tau+1}'$. The symmetry ϕ on S is represented by reflection in C; the τ transition curves on S by C and the $\tau - 1$ pairs of circles Γ_s, Γ_s'. The pairs of quasicircles Λ_i, Λ_i' represent (when identified under reflection) $g - \tau + 1$ cross caps on S/ϕ.*

6. **Fixed point free diasymmetric surfaces.** To extend the results of §§4 and 5 to symmetric surfaces of type $(g, 0)$ we use a different representation of the symmetry. This is clearly necessary, since the reflection $R_0(z)$ always leaves fixed the points on the unit circle.

Let $K_0, \cdots, K_g$ be disjoint circles and let $Q_j(z)$, $0 \leq j \leq g$, be reflection in K_j followed by rotation about the center of K_j by the angle π. Denote by K_j' the circle $Q_0(K_j)$, $1 \leq j \leq g$. Denoting by F the exterior of the $2g$ circles $K_1, K_1', \cdots, K_g, K_g'$ we see that F is a standard fundamental domain of the Schottky group $G = \{Q_0 \circ Q_1, \cdots, Q_0 \circ Q_g\}$. F/G is a symmetric surface (with respect to the symmetry Q, $Q \circ \pi = \pi \circ Q_0$). It has genus g and no transition curves. Since, for a point P on K_r, $Q_0(P)$ and $Q_0 \circ Q_r(P)$ are diametrically opposed points on K_r'. $(F/G)/Q = F/\{G, Q_0\}$ is a sphere with $g + 1$ cross caps. If S is a symmetric surface of type $(g, 0)$ there exists a homeomorphism f of F/G onto S such that $\phi \circ f = f \circ Q$.

As in §§4 and 5 we obtain a fundamental domain F^μ of a Schottky group G^μ which represents S. Furthermore,

(a) The symmetry ϕ on S is represented by a symmetry $Q_0^\mu(w)$ which is of the form (see Lemma 3)

$$Q_0^\mu(w) = b - \lambda^2/\overline{w - b}.$$

(b) Again by Lemma 3, K_j^μ and $K_j'^\mu$ are quasicircles and the Möbius transformation $(Q_0 \circ Q_j)^\mu(w)$ maps the exterior of K_j^μ onto the interior of $K_j'^\mu$ in such a way that each point P on one quasicircle is identified with that point P'' on the other quasicircle which is diametrically opposed to the point $Q_0^\mu(P)$. Therefore, points on F^μ which lie on the quasicircles are identified, under the group $\{G^\mu, Q_0^\mu\}$ with their diametrically opposed points.

We assume without loss of generality that the "center" of K_0^μ is at the origin and that $\lambda = 1$, so that $Q_0^\mu(w) = -1/\bar{w}$. We can then state

THEOREM III. *A symmetric Riemann surface S of type $(g, 0)$ with respect to a symmetry ϕ can be represented by a Schottky group G having a fundamental domain bounded by g identified pairs of disjoint quasicircles $\Lambda_1, \Lambda_1', \cdots, \Lambda_g, \Lambda_g'$ which are symmetrically situated with respect to the symmetry $\tilde{Q}(w) = -1/\bar{w}$. The symmetry ϕ on S is represented by the transformation $\tilde{Q}(w)$. There is also a quasicircle Λ which is a closed Jordan curve and which contains in its interior the quasicircles $\Lambda_1, \Lambda_2, \cdots, \Lambda_g$. $\tilde{Q}(w)$ transforms Λ into itself in such a way that diametrically opposed points are identified, and transforms Λ_i into Λ_i' in such a way that a point P on Λ_i is identified with the point on Λ_i' which is diametrically opposed to the point on Λ_i' which is identified with P under the group G. As a result, the quasicircle Λ, together with the g pairs of quasicircles Λ_i, Λ_i' represent, when identified under $\tilde{Q}$, $g+1$ cross caps on S/ϕ.*

7. **Mappings of multiply connected domains.** We recall that a multiply connected plane domain D, bounded by n nondegenerate continua, can be mapped conformally onto a plane domain bounded by n closed analytic Jordan curves. This follows at once from the Riemann mapping theorem.

Given a domain D bounded by n closed analytic Jordan curves $\gamma_1, \cdots, \gamma_n$, let S be the closed surface obtained by doubling D [2, pp. 118-119]. We observe that S is a Riemann surface of genus $n-1$, orthosymmetric with respect to the symmetry ϕ defined by the doubling process. Furthermore $S/\phi = D$. Then by Theorem I, D is conformally equivalent to a region bounded by n disjoint circles. The above arguments give a new proof of the

KOEBE THEOREM. *A multiply connected plane domain, bounded by n nondegenerate continua can be mapped conformally onto a plane domain bounded by n circles.*

REMARK. This theorem, which has been obtained as a corollary of Theorem I, can also be obtained directly from Lemma 2.

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